Minimum Weight Design with Structural Reliability

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A procedure is proposed for designing a structure with minimum weight and desired reliability. Simple techniques are presented for the determination of the desirable probabilities of failure, the resulting factors of safety, and the minimum weight designs of the components of the structure. An elementary analysis indicates that the over-all minimum weight is approached when the individual probabilities of failure of the components of the structure are made proportional to their weight. Probabilities of failure are thus established and transferred to equivalent factors of safety based upon the distributions (mean values and variabilities) of the applied and failing stresses. These can be determined by the distributions of the independent random variables and from criteria such as static tension, creep, fatigue, and stability, which are assumed or established from test data. Minimum weight designs are then determined by available techniques. A simple iterative scheme is employed, since the weights of the components have to be determined. The design techniques are illustrated by a simple truss example.

Nomenclature

a = ratio of coefficients of variability (γ_a/γ_f)
b = width
d_0 = dimension
d = diameter
l = length
n = number of components or tests
r = factor of safety ($\bar{x}_f/\bar{x}_a$)
t = thickness
u = nondimensional deviation from the mean $[(y - \bar{y})/\sigma_y = \phi^{-1}(u)]$
x = stress
$\bar{x}$ = mean stress value
y = failing and applied stress difference ($x_f - x_a$)
A = area
C = stability constant
D = damage parameter
E = modulus of elasticity
F = axial load
P = design load (rF)
R = radius
R = test scatter factor
W = weight
γ = coefficient of variability ($\sigma_x/\bar{x}$)
δd = dimensional tolerance
λ = Lagrange constant
ξ = deviation parameter ($u\gamma_f$)
ρ = density
σ = deviation
$\phi(u)$ = cumulative standardized normal distribution function $\left[\left(\frac{1}{2\pi}\right)^{1/2}\int_{-\infty}^{u} e^{-u^2/2}du\right]$
ϕ_{-} = probability of failure
$\bar{(\ )}$ = over variables represents mean value

Subscripts

a = applied
c = confidence level
f = failure
fA = A values of Ref. 4
fB = B values of Ref. 4
i, j, k = components

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The minimum weight design of a structure requires a knowledge of the thermal and mechanical environmental history as well as the applicable failure modes for the component members. The design must also incorporate a factor of safety which will insure the required structural reliability.

This paper presents a method for designing component members of a minimum weight structure that will have a required over-all reliability consistent with specific predetermined requirements. A structural component is defined as a member whose behavior can be predicted by available analysis and representative tests. Size as well as volumetric effects are assumed to be adequately represented by allowables obtained from tests. The procedure is to design each component member with a properly proportioned probability of failure to achieve minimum weight and required over-all reliability. The design technique utilizes relationships that determine the equivalent factors of safety needed to design each component using standard design procedures. The factor of safety is dependent upon the probability of failure and the variability of the applied and failing stresses, which are readily determined or estimated from given data. The failing stresses employed must correspond to the criteria governing the design.

The design procedure can be employed with any type of distribution, since the equations presented are applicable to all. The distribution employed should be that one which the designer believes is applicable to the structural problem being studied. A normal (Gaussian) distribution is recommended for the purpose of obtaining numerical results. The Gaussian distribution results in a simplification of the computation by permitting the use of available technology (mathematical relationships[1] and tabular data[2]). It usually results in a satisfactory approximation of the distribution of the structural variables and can readily be adapted to limited sampling data. It should not be applied, obviously, to exceedingly small probabilities where it may present impossible values (e.g., negative dimensions).

I. Component Probabilities

A permissible probability of failure of the complete structure is usually prescribed after consideration is given to the general requirements and costs of the system. The over-all structural reliability may be considered as a function of the reliability of the structural components. These components may be visualized as springs in series and parallel to the applied loads. For the purpose of analysis, one could consider the structure as composed of a set of series elements, since each subset of parallel elements can be transformed into an equivalent series element[1]

$$\left(\phi_i = \prod_{i=1}^{n} \phi_{i,j}\right)$$

Failure of a parallel element results in a redistribution of the load among the remaining parallel elements of that subset, whereas failure of a series element results in failure of the structure.

The probability of failure for a structure composed exclusively of elements in series (e.g., a statically determinate structure) can be expressed as

$$\phi = 1 - \prod_{i=1}^{n} (1 - \phi_i) \tag{1a}$$

The values of ϕ_i are always smaller than ϕ, which is usually quite small. A good approximation of ϕ is therefore given by

$$\phi = \sum_{i=1}^{n} \phi_i \tag{1b}$$

which is obtained by neglecting products of ϕ_i in Eq. (1a).

The weight of the structure (W) is the sum of the weights of the individual elements, expressible as

$$W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \rho_i l_i A_i \tag{2}$$

The problem is to minimize W, subject to the constraint condition defined by Eq. (1b). This is best accomplished by the use of LaGrange's method of an undetermined multiplier (λ), which minimizes a function equal to the weight, augmented by a zero term expressing the constraint condition. The partial derivates of this function with respect to the weight of the elemental members are set equal to zero. This results in

$$(\partial/\partial W_j)[W + \lambda(\phi - \Sigma\phi_i)] = 1 - \lambda\partial\phi_j/\partial W_j = 0 \tag{3a}$$

$$1/\lambda = \partial\phi_j/\partial W_j \tag{3b}$$

where W and ϕ are stationary, and where it is assumed that the probability of failure for the jth component is independent of the weight of the kth component ($\partial\phi_j/\partial W_k = 0$). This condition is satisfied by a statically determinate structure.

It is implied in Eq. (3b) that at over-all minimum weight changes in the probability of failure of each component are proportional to its change in weight, and that this ratio is independent of the respective components. This can be seen intuitively, since the weight of a component increases with a decrease in the probability of failure. Thus the over-all probabilities of failure should be distributed among the members of the assembly, so that the heavier members receive a greater proportion than the lighter members.

For the case of a nonminimum over-all weight structural design, increasing the probability of failure of the heavier members while correspondingly decreasing the probability of failure of the lighter members, in order to maintain the over-all probability of failure, tends to cause a decrease in weight of the heavier ones which would be greater than the corresponding increase in weight of the lighter components. This would result in an over-all decrease in weight.

For the case of minimum over-all weight designs, the exact functional relationship of ϕ_i to W_i is not clearly evident. It is further assumed, therefore, that the ratio of the weight of a component to the over-all weight is relatively insensitive to the over-all probability of failure. This is equivalent to

$$(W_i/\Sigma W_i)_{\phi(u_1)} = (W_i/\Sigma W_i)_{\phi(u_2)} \tag{4a}$$

This relationship is implied in the regime of structural design where the probability of failure is small. A small proportional change in the weight of a component produces a much larger proportional change in the probability of failure. Thus, small changes in the over-all probability of failure should result in insignificant changes in the weight ratio (see Sec. VI D, "Illustrative Example").

Equations (3b) and (4a) are satisfied if

$$\phi_i = (\phi/W)W_i \tag{4b}$$

This suggests that a minimum weight design would result when the probability of failure of each component member is proportional to its weight. The relative simplicity of Eq. (4b) is advantageous in obtaining the optimum component probability of failure as a function of weight, and it is desirable in the design procedure, because the over-all weight is unknown and the final design must be determined by an iteration process. The relationship has the correct sense, is obviously correct for identical elements, and degenerates satisfactorily to the case of a single component.

If the optimum probability of failure is known, then it is necessary to obtain a minimum weight design consistent with this value. This requires the determination of a failing to applied load ratio (factor of safety), as well as a minimum weight design procedure for a given failing load.

II. Factor of Safety

The structure must be designed for a failing load that results in the desired probability of failure when subjected to the expected load-temperature history. The probability of failure for the two stress distributions x_a and x_f is equal to the probability that $(x_f - x_a) \leq 0$. This probability can be obtained as follows. Let

$$y = x_f - x_a \tag{5a}$$

Then from Ref. 1 we obtain

$$\bar{y} = \bar{x}_f - \bar{x}_a \tag{5b}$$

$$\sigma_y = (\sigma_f^2 + \sigma_a^2)^{1/2} \tag{5c}$$

and the probability that y is less than zero is

$$\phi(u) = \phi(-\bar{y}/\sigma_y) = \phi([\bar{x}_a - \bar{x}_f]/[\sigma_f^2 + \sigma_a^2]^{1/2})$$

Therefore

$$u = (\bar{x}_a - \bar{x}_f)/(\sigma_f^2 + \sigma_a^2)^{1/2} \tag{5d}$$

Solving for $r = \bar{x}_f/\bar{x}_a$ in Eq. (5d) gives

$$r = \frac{\bar{x}_f}{\bar{x}_a} = \frac{1 + |\xi|((1 + a^2) - a^2\xi)^{1/2}}{1 - \xi^2} \tag{6a}$$

where

$$\xi = u\gamma_f \tag{6b}$$

$$a = \gamma_a/\gamma_f \tag{6c}$$

Equation (6a) is plotted in Fig. 1. Values of u equal to the inverse of the probability of failure [$\phi(u)$] are obtained from a table of the Cumulative Standardized Function, which is available in various mathematical texts.[2]

The quantities necessary to utilize Eq. (6) are the mean values and the variabilities of the applied and failing stresses. They are readily determined from available data (e.g., Ref. 4, load criteria, dimensional tolerance, experimental data, etc.). This can be illustrated by the following examples.

1) The A and B values of the failing stress in Ref. 4 are given:

A = $x_{fA} \leq 99\%$ of the values of x_f

$[\phi(u) = 0.01;\ u = -2.327]$

B = $x_{fB} \leq 90\%$ of the values of x_f

$[\phi(u) = 0.1;\ u = -1.287]$

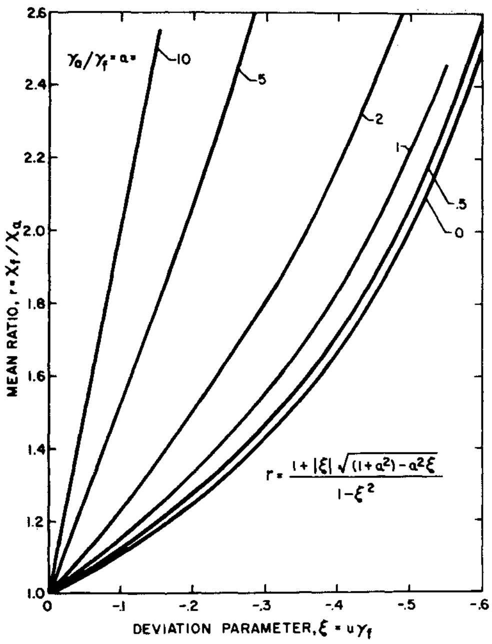

Fig. 1 Factor of safety as a function of probability and variability.

then

$$\bar{x}_f = B + 1.237(B - A) \tag{7a}$$

$$\gamma_f = \sigma_f/\bar{x}_f = (B - A)/1.04\bar{x}_f \tag{7b}$$

2) Given the dimension $d_0 \pm \delta d$, and assuming that the tolerance corresponds to three deviations, results in

$$\bar{d} = d_0 \tag{8a}$$

$$\gamma_d = \delta d/3d_0 \tag{8b}$$

Similarly, the mean value and variability of the dependent variables are obtained by a transformation of the independent variables.[1] As an example, for a product of several variables, if

$$x = \alpha\beta\delta \ldots \tag{9a}$$

then

$$\bar{x} = \bar{\alpha}\bar{\beta}\bar{\delta} \ldots \tag{9b}$$

$$\gamma_x^2 = \gamma_\alpha^2 + \gamma_\beta^2 + \gamma_\delta^2 \ldots \tag{9c}$$

III. Failure Criteria

The design must consider all possible modes of failure in order to assure structural adequacy in the critical mode. Among the modes of failure that should be considered are static tension, compressive instability, fatigue, and excessive deformations.

When the failing mode depends upon a material property that is known, such as the tensile stress tabulated in Ref. 4, then the mean and the variability of the failing stress are determined as indicated in Eqs. (7a) and (7b) and illustrated in Sec. IV A1, "Minimum Weight Design."

In many cases, the failure stress population is not known, but must be estimated from limited experimental samples. For short time strengths, data from available or additional tests on structural models can be analyzed to determine the mean and variability of significant variables. As an example, tests of monocoque cylinders in compression can be analyzed to determine the distribution of the stability coefficient. Thus, from Eqs. (9b) and (9c),

$$\bar{C} = \bar{x}/\bar{E}\bar{t}(\overline{1/R}) \tag{10a}$$

$$\gamma_C = (\gamma_f^2 - [\gamma_E^2 + \gamma_t^2 + \gamma_{1/R}^2])^{1/2} = \sigma_C/\bar{C} \tag{10b}$$

In other instances, failure is dependent upon a load and temperature history as exemplified by fatigue or creep deformation. In this case, the material behavior as evidenced by available tests can be employed with an appropriate cumulative damage theory ($D^* = \Sigma D_i$) to obtain the necessary design information. The test data can then be analyzed to evaluate the value of the mean ($\bar{D}^*$) and variability (γ_{D*}) of the damage that causes failure. The given load and temperature history is then employed with the variability of area and the distribution of the damage criteria to obtain an applied stress history which would result in a calculated cumulative damage, with the desired probability of failure at the end of the required life. A similar procedure is employed for maximum creep strain in illustrative example B of Ref. 3. Confidence in designs employing cumulative damage failure criteria should be obtained from simulated tests on the actual design. The confidence level can be estimated from equations of the following type:[1]

$$(n^{1/2}/\gamma_{D*})(R - 1) \leq u_c \tag{10c}$$

where n = number of tests, and R = test, scatter factor (e.g., test damage/design damage parameter ratio = $D_{\text{test}}^*/D_{\text{design}}^*$).

IV. Minimum Weight Design

Having established a desired probability of failure, a design criteria, and a required factor of safety, it is then necessary to determine a minimum weight design for each element. When the design criteria are not defined, then several design criteria should be examined to determine which is critical. Minimum weight design techniques, which can be readily coupled with the factor of safety, are given in Ref. 3.

Designs can be grouped in two types. In the first, the design failing stress can be obtained as a function of the material and load history. This is exemplified by designs that are governed by maximum stress considerations such as static tension, fatigue, creep, excessive deformation, and by other designs that are independent of the detail geometry (e.g., honeycomb sandwich). In the second, the failing stress is dependent upon the geometric details as exemplified by compressive stability.

Examples of design equations for structural elements of constant cross section are presented below. The equations are presented in terms of the mean values of the random variables, with the bars not shown to simplify the presentation. It should be noted that the value of r is determined from Eq. (6a) once the probability of failure $\phi(u)$ and the variabilities (γ_f and γ_a) are established for a structural component. Thus, changes in the mean value of the applied load with constant $\phi(u)$, γ_f, and γ_a result in proportional changes in the design dimensions as defined in the equations below.

A. Known Design Stress

1. *Static strength*

The cross-sectional area is determined from

$$A = rF/x_f \tag{11}$$

The design technique of an element can be illustrated by the following example. Design a 7075-T6 aluminum sheet with tolerance of 10% in thickness, for a probability of failure of 0.01 when subjected to a tension load of $\bar{F}$ = 7000 lb/in. and a coefficient of variability $\gamma_F = \sigma_F/\bar{F} = 0.05$.

From Ref. 4, it is found that $A = x_{fA}$ = 76,000 psi, and $B = x_{fB}$ = 78,000 psi. From Eqs. (7a) and (7b),

$$\bar{x}_f = 78{,}000 + 1.237\ (78{,}000 - 76{,}000) = 80{,}474 \text{ psi}$$

and

$$\gamma_f = (78{,}000 - 76{,}000)/1.04(80{,}474) = 0.024$$

From Eqs. (8b) and (9c),

$$\gamma_a = [(0.05)^2 + (0.10/3)^2]^{1/2} = 0.06$$

Therefore,

$$a = \gamma_a/\gamma_f = 0.06/0.024 = 2.5 \quad (6c')$$

For $\phi(u) = 0.01$ we obtain, from Ref. 2, $u = -2.327$; therefore,

$$\xi = -2.327(0.024) = -0.054 \quad (6b')$$

Substituting in Eq. (6a) or from Fig. 1, we obtain $r = 1.15$. The mean applied stress is

$$x_a = x_f/1.15 = 80474/1.15 = 69{,}990 \text{ psi}$$

Therefore,

$$t = F/x_a = \frac{7000}{69900} = 0.100 \text{ in.}$$

It is interesting to note that a design based upon a low estimate of the failing stress, which is approximately 1.5 times a high estimate of applied stress, would result in an exceedingly low probability of failure when the applied and failing stress variabilities are not excessive (see Table 4 of Ref. 1).

2. *Cumulative damage*

The given load-temperature history, variability of area, and the analysis of the test data employing cumulative damage theory, result in a maximum stress level that has the desired probability of failure, i.e.,

$$D_{\text{test}}^*/r(u, \gamma_f, \gamma_a) = D_{\text{design}}^* = D^*(x_a)$$
$$A = F/x_a \quad (12)$$

B. Unknown Design Stress

The general design technique permitting inelastic as well as elastic design stresses is presented in Ref. 3 where the design load is made equal to the mean applied load times the factor of safety ($P = rF$). The design requires the solution of nonlinear design equations.[3] The solutions are presented graphically[3] and degenerate to the following equations for elastic stresses.

1. *Compressive instability with one unknown*

The design is obtained by expressing the area and stability stress in terms of the unknown dimension.

a) Thickness of plate (t): Equating the applied stress to the instability stress modified by the factor of safety results in

$$F/tb = x_a = x_f/r = (C_tE/r)(t/b)^2 \quad (13a)$$

therefore,

$$t = (rFb/C_tE)^{1/3} \quad (13b)$$

b) Thickness (t) of a constant radius (R) cylinder: A similar procedure results in

$$F/2\pi Rt = x_a = x_f/r = CE(t/R)/r \quad (14a)$$

therefore,

$$t = (rF/2\pi CE)^{1/2} \quad (14b)$$

2. *Compressive instability with two unknowns*

The design technique is exemplified by a pin-ended tubular column. The design procedure equates the instability stresses and the applied stress. This results in

$$\frac{F}{2\pi Rt} = x_a = \frac{x_f}{r} = \frac{CE}{r}\left(\frac{t}{R}\right) = \frac{\pi E}{rl^2}\frac{I}{A} = \frac{\pi E}{2r}\left(\frac{R}{l}\right)^2 \quad (15a)$$

Therefore,

$$R = (2Cl^4Fr/\pi^5E)^{1/6} \quad (15b)$$
$$t = (rF/2\pi CE) \quad (15c)$$

Similar design equations can be formulated for other types of constructions.[3]

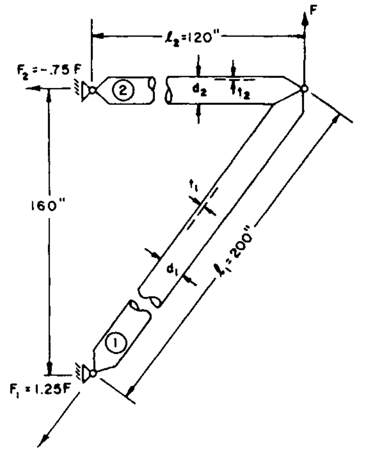

Fig. 2 Loaded structure.

V. Design Procedure

The weight distribution between the component members is unknown and must be assumed in order to obtain an initial estimate of the optimum probabilities of failure from Eq. (4b). Considering the failure criteria and the variability of the applied and failing stress, a factor of safety is determined from Eq. (6) and is employed in the appropriate minimum weight design procedure to obtain a design and resulting component weights. A significant difference between the calculated and assumed weight ratios would indicate a repetition of the design procedure with a better estimate of the weight ratios. One iteration is usually sufficient, since the design weight is not too sensitive to small variations in the structural probabilities of failure. The simplicity of this design technique suggests that a digital program can be readily formulated to perform the iterations that may be required when the structural problem becomes complex (e.g., large number of members).

VI. Illustrative Example

The design procedure can be illustrated by the following simple example, which serves to illustrate an iterative scheme that can be presented in tabular form with a minimum of explanation. A Gaussian distribution of the structural variables was assumed in performing the numerical calculations.

The problem is to determine the constant wall thickness (t_1 and t_2) of the structure shown in Fig. 2.

Given

$\bar{F} = 100{,}000$ lb

$\bar{F}_1 = 125{,}000$ lb $\qquad \gamma_F = 0.04$

$\bar{F}_2 = -75{,}000$ lb

$\bar{d}_1 = \bar{d}_2 = 10$ in.

$\gamma_d = 0.01$

$\gamma_t = 0.02$

$$\bar{C} = \frac{\bar{x}_f}{2\bar{E}\bar{t}(1/d)} = 0.4 \qquad \gamma_C = 0.15 \text{ (assumed)}$$

material 7075-T6 at room temperature, thus

$$x_{fA} = 76{,}000 \text{ and } x_{fB} = 78{,}000 \text{ (Ref. 4)}$$

and

$$\bar{E} = 10^7 \text{ psi} \qquad \gamma_E = 0.01 \text{ (assumed)}$$

To Find

$\bar{t}_1$ and $\bar{t}_2$ for $\phi = 0.1$, 0.01, and 0.001

Table 1 Design of structure

ϕ	0.1			
Assumed weight, W_1/W_2	1.00		1.33[a]	
Component	Tube 1[b]	Tube 2[b]	Tube 1[b]	Tube 2[b]
Variability[b]	$\gamma_f = 0.024$ $a = 1.92$	$\gamma_f = 0.152$ $a = 0.303$		
Optimum probability $\phi_i = \phi \frac{W_i}{\Sigma W_i}$	0.05	0.05	0.0571	0.0429
Deviation u_i (Ref. 2)	−1.645	−1.645	−1.58	−1.718
$\xi_i = u_i\gamma_f$	−0.0394	−0.250	−0.0379	−0.261
Safety factor r_i (Fig. 1)	1.08	1.35	1.075	1.37
Calculated weight ratio $\frac{1.45r_1}{r_2^{1/2}}$	$1.345 \neq 1.00$		$1.33 = 1.33$	
Thickness, $\bar{t}_i$ Eqs. (10) and (14b)	...	...	0.0532	0.0637

ϕ	0.01				0.001			
W_1/W_2	1.0		1.25[a]		1.0		1.18[a]	
Component	Tube 1[b]	Tube 2[b]	Tube 1[b]	Tube 2[b]	Tube 1[b]	Tube 2[b]	Tube 1[b]	Tube 2[b]
φ_i	0.005	0.005	0.00555	0.00445	0.0005	0.0005	0.000541	0.000459
u_i	−2.575	−2.575	−2.54	−2.616	−3.29	−3.29	−3.267	−3.315
ξ_i	−0.062	−3.92	−0.061	−0.397	−0.079	−0.500	−0.0785	−0.504
r_i	1.13	1.68	1.12	1.685	1.17	2.035	1.17	2.05
$1.45r_1/r_2^{1/2}$	$1.265 \neq 1.00$		$1.25 = 1.25$		$1.19 \neq 1.00$		$1.18 = 1.18$	
$\bar{t}_i$	...	...	0.0555	0.0708	...	...	0.0578	0.0781

[a] This value is a better assumption of weight distribution that is close to the calculated value, but modified in the direction of original assumption.
[b] The variability (γ_f and γ_a) for each tube does not change.

A. Design Equations

Member 1 is designed for static tension; therefore,

$$\bar{t}_1 = r_1\bar{F}_1/\pi\bar{d}\bar{x}_{f1} \tag{11'}$$

with $\bar{x}_{f1} = 80474$ and $\gamma_{f1} = 0.024$ (see example in Sec. IV A1).

$$\gamma_{a1} = \gamma_{a2} = [\pi_F^2 + \gamma_{1/d}^2 + \gamma_{1/t}^2]^{1/2}$$
$$= [(0.04)^2 + (0.01)^2 + (0.02)^2]^{1/2} \tag{9c'}$$
$$= 0.046$$

and

$$a_1 = \gamma_{a1}/\gamma_{f1} = 0.046/0.024 = 1.92 \tag{6c''}$$

Member 2 is designed for local instability of the tube, since the tube is not critical in over-all buckling. The axial load was made sufficiently small, so that the buckling stress would be elastic. Hence, since

$$\frac{\bar{F}}{\pi\bar{d}\bar{t}} = \frac{\bar{x}_f}{r} = \frac{2\bar{C}\bar{E}\bar{t}\overline{(1/d)}}{r}$$

we obtain

$$\bar{t}_2 = (r_2\bar{F}_2/2\pi\bar{C}\bar{E})^{1/2} \tag{15c'}$$

Member 2 has the following variabilities:

$$\gamma_{f2} = [\gamma_C^2 + \gamma_E^2 + \gamma_t^2 + \gamma_{1/d}^2]^{1/2}$$
$$= [(0.15)^2 + (0.01)^2 + (0.02)^2 + (0.01)^2]^{1/2} \tag{10b'}$$
$$= 0.152$$

and

$$a_2 = \gamma_{a2}/\gamma_{f2} = 0.046/0.152 = 0.303 \tag{6c'''}$$

B. Weight Comparison

The weight ratio for this problem can be established in terms of the factors of safety, since

$$\frac{W_1}{W_2} = \frac{\rho_1 l_1 \pi d_1 t_1}{\rho_2 l_2 \pi d_2 t_2} = \frac{\pi\rho_1 l_1 d_1(r_1F_1/\pi d_1 x_{f1})}{\pi\rho_2 l_2 d_2(r_2F_2/2\pi CE)^{1/2}} \tag{16a}$$
$$= \frac{\rho_1}{\rho_2}\frac{l_1}{l_2 d_2}\frac{(r_1F_1/x_{f1})}{(\pi r_2 F_2/2CE)^{1/2}}$$

Substituting the known data into Eq. (16a) results in

$$W_1/W_2 = 1.45(r_1/r_2^{1/2}) \tag{16b}$$

C. Design Tables

The problem is solved in tabular form by assuming a weight ratio to calculate probabilities of failure, factors of safety, and resulting design. This procedure is continued until the calculated weight ratio is sufficiently close to the assumed ratio. The results are shown in Table 1.

D. Discussion

The calculations confirm the hypothesis that the over-all weight increases with a decrease in the probability of failure [$W_{0.01}/W_{0.1} = 1.064$ and $W_{0.001}/W_{0.1} = 1.126$]. The calculations also indicate a rapid convergence of the iterative scheme, with the optimum weight ratio obtained in one or two cycles. The relative insensitivity of the design weights, resulting from the initial and final weight ratio estimates, are evidenced by the small changes in r. This would indicate that the exact optimum probability of failure is not too critical. The weight ratios for the initial and final designs ($W_{0.1,\ 1}/W_{0.1,\ 1.33} = 1.0020$, $W_{0.01,\ 1}/W_{0.01,\ 1.25} = 1.0043$, $W_{0.001,\ 1}/W_{0.001,\ 1.18} = 1.000$) would indicate relatively insensitive weight functions in these design regions, with the lightest weight in the vicinity of the probability defined by Eq. (4b). This example indicated small changes in the optimum weight ratio

$$[(W_1/\Sigma W)_{0.1}/(W_1/\Sigma W)_{0.01}/(W_1/\Sigma W)_{0.001} = 0.57/0.56/0.55]$$

for orders of magnitude changes in the over-all probability of failure, even though relatively high values of ϕ were employed. The weight ratios should be even less sensitive for the usually small values of ϕ. Thus the design procedure based on Eq. (4b) should be satisfactory.

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